Optimal control:

Additional condition: U, and X are metric spaces, and U is compact.

$f : X \times U \to X$ and $L : X \times U \to \mathbb{R}$ are continuous.

Proposition: Assume that the additional conditions hold true.

① T defined by $T(h)(x) := \inf_{u_0 \in U} \left(L(x, u_0) + \beta h(f(x, u_0)) \right)$ is also contracting from $C_b(X) := \{$ bounded continuous function on X. $\}$ to $C_b(X)$

② The unique solution of $V = T \circ V$ satisfies $\boxed{V \in C_b(X)}$ and there exist $u^* : X \to U$ s.t.

$$\inf_{u_0 \in U} \left(L(x, u_0) + \beta V(f(x, u_0)) \right) = L(x, u^*(x)) + \beta V(f(x, u^*(x)))$$

for all $x \in V$.

③ Let $\bar{X}_0 := x_0$, $\bar{u}_n := u^*(\bar{X}_n)$, $\bar{X}_{n+1} = f(\bar{X}_n, \bar{u}_n)$ $\forall n \geq 0$.

then $\bar{u}$ is an optimal control to the (infinit horizon) control problem.

i.e. $V(x_0) = \inf_{u} \left(\sum_{n=0}^{+\infty} \beta^n L(X_n^{0, x_0, u}, u_n) \right) = \sum^{+\infty} \beta^n L(\bar{X} \ldots$

$(u_n)_{n\geq 0}$ ... $\sum_{n=0} \beta^n L(x_n, u_n)$.

Proof: ① Let $h \in C_b(X)$, we will prove that $T(h) \in C_b(X)$

Let $\ell(x,u) := L(x,u) + \beta h(f(x,u))$ so that ℓ is bounded continuous in (x,u).

Then $T(h)(x) = \inf_{u \in U} \ell(x,u)$ is continous in x since U is compact.

($x_n \to x$. (u_n) ....)

Besides, we already know that $\|T(h_1) - T(h_2)\|_\infty \leq \beta \, \|h_1 - h_2\|_\infty$

② Since $(C_b(X), \|\cdot\|_\infty)$ is a complet space.

Then. $V = T(V)$ has a unique solution in $C_b(X) \subset B(X)$

i.e. $V \in C_b(X)$

And by the continuity of $(x,u) \mapsto L(x,u) + \beta V(f(x,u))$

and the compactness of U.

there exists $u^*: X \to U$. s.t $T(V)(x) = L(x, u^*(x)) + \beta V(f(x, u^*(x)))$.

③ We claim that $V(x_0) = \sum^{N-1} \beta^n L(\bar{x}_n, \bar{u}_n)$ ... β^N ...

$$\textcircled{*} \quad \underline{\qquad} \quad \sum_{n=0} \beta^n L(\bar{x}_n, \bar{u}_n) + \beta^N V(\bar{x}_N), \text{ for all } N \geq 0.$$

Then let $N \to +\infty$, one obtains that $V(x_0) = \sum_{n=0}^{+\infty} \beta^n L(\bar{x}_n, \bar{u}_n)$.

For claim, we will use the induction argument.

First, $\textcircled{*}$ is true for $N = 0$.

Next, assume that $\textcircled{*}$ holds for N.

$$\text{So} \quad V(x_0) = \sum_{n=0}^{N-1} \beta^n L(\bar{x}_n, \bar{u}_n) + \beta^N V(\bar{x}_N)$$

By DP: $V(\bar{x}_N) = T(V)(\bar{x}_N) = L(\bar{x}_N, u^*(\bar{x}_N)) + \beta V(\underbrace{f(\bar{x}_N, u^*(\bar{x}_N))}_{= \bar{x}_{N+1}})$

Then, $V(x_0) = \sum_{n=0}^{N-1} \beta^n L(\bar{x}_n, \bar{u}_n) + \beta^N \cdot L(\bar{x}_N, u^*(\bar{x}_N))$

$$+ \beta^N \beta V(\bar{x}_{N+1})$$

$$= \sum_{n=0}^{N} \beta^n L(\bar{x}_n, \bar{u}_n) + \beta^{N+1} V(\bar{x}_{N+1}).$$

i.e. $\textcircled{*}$ holds for $N+1$.

#

x_n

dynamic

Deterministic setting: $X_{n+1} = f(X_n, u_n)$ dynamic, $\sum_{n=0}^{+\infty} L(X_n, u_n)$ cost function

Stochastic setting: $\mathcal{X} := \{1, \dots, J\}$, $U := \{u_1, \dots, u_I\}$

$$\mathbb{P}\left[X_{n+1}^{u_k} = j \mid X_n^{u_k} = i, u_k\right] = p(i, j, u_k) \quad \forall\, i, j \in \mathcal{X},\ k = 1 \dots I.$$

Value function:

$$V(k, x) := \inf_{u} \mathbb{E}\left[\sum_{n=k}^{N-1} L(X_n^{k,x,u}, u_n) + g(X_N^{k,x,u})\right]$$

$(X_k^{k,x,u} = x.)$

or.

$$V(x) := \inf_{u} \mathbb{E}\left[\sum_{n=0}^{+\infty} \beta^n \cdot L(X_n^{k,x,u}, u_n)\right]$$

Admissible control: u_n is a function of $(X_0, \dots, X_n)$, $\forall n$

Dynamic programming: ① Finite horizon.

$$V(n, x) = \inf_{u_n \in U} \mathbb{E}\left[L(x, u_n) + V(n+1, X_{n+1}^{n,x,u_n})\right].$$

(2) Infinite horizon: $V(x) = \inf_{u_0 \in U} \mathbb{E}\left[L(x, u_0) + \beta V(X_1^{0,x,u_0}) \right]$

Remark: - The proof is similar to the deterministic setting.

- In practice, we estimate $\mathbb{E}[\xi]$ in place of computing $\mathbb{E}[\cdot]$ by $\frac{1}{K}\sum_{k=1}^{K} \xi_k$

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Exercise 1. $W(x) := \inf \left\{ \sum_{k=0}^{N-1} u_k^2 : u_k \geq 0 \text{ and } \sum_{k=0}^{N-1} u_k = x \right\}$

$\Leftrightarrow -u_k \leq 0$

$k = 0, \cdots, N-1$

$x > 0$.

$\inf \{ f(u) : g(u) \leq 0,\ h(u) = 0 \}$

(Solution is $u_k = \frac{x}{N}$. $\Rightarrow W(x) = \frac{x^2}{N}$.)

$f(u) = \sum u_k^2$ / $g_k(u) = -u_k$, $h(u) = \sum u_k - x$.

Static approach: ① Give the necessary condition from the KKT theorem.

② Solve the problem with the necessary condition.

- Existence of optimal solution. / qualification.

- $L(u, \lambda_0, \cdots, \lambda_{N-1}, \mu) = \sum_{k=0}^{N-1} u_k^2 + \sum_{k=0}^{N-1} \lambda_k (-u_k) + \mu \left(\sum_{k=0}^{N-1} u_k - x \right)$

$\exists\ \lambda \in \mathbb{R}^N_+,\ \mu \in \mathbb{R}$. s.t.

① $\partial_{u_k} L(u, \lambda, \mu) = 2u_k - \lambda_k + \mu = 0. \quad \forall\ k = 0, \dots, N-1.$

② $\sum_{k=0}^{N-1} \underbrace{\lambda_k}_{\geq 0} (-\underbrace{u_k}_{\geq 0}) = 0 \iff \forall\ k,\ \lambda_k u_k = 0 \quad \left(\sum_{k=0}^{N-1} u_k = x \right)$

If $u_k \neq 0 \overset{②}{\Longrightarrow} \lambda_k = 0 \overset{①}{\Longrightarrow} u_k = \mu/2 \quad \forall\ k = 0, \dots, N-1$

Assume that there are N_0 terms in $(u_k)_{k=0,\dots,N-1}$ s.t. $u_k \neq 0$.

$\Longrightarrow \sum_{k=0}^{N-1} u_k = N_0\, \mu/2 = x. \Longrightarrow \mu = \frac{2x}{N_0}$

$\Longrightarrow W(x) = \sum u_k^2 = N_0 \frac{\mu^2}{4} = N_0 \frac{4x^2}{4N_0^2} = \frac{x^2}{N_0}$

If $N_0 < N$ then $W(x) = \frac{x^2}{N_0} > \frac{x^2}{N} \Longrightarrow \mu = \frac{2x}{N_0}$ and $u_k = \frac{x}{N_0}$ is not optimal.

So. $N_0 = N$, and $u_k = \frac{x}{N}$ is the optimal solution.

$\Longrightarrow W(x) = \frac{x^2}{N}$. #

Dynamic approach:

$X_0 = x.$ $\quad X_{n+1} = X_n - u_n,$ $\quad L(x,u) = u^2.$

$u_n \in U_n := [0, X_n]$

$$\inf_u \sum_{n=0}^{N-2} L(X_n, u_n) + X_{N-1}^2 \iff \inf_u \left(\sum_{k=0}^{N-2} u_k^2 + X_{N-1}^2 \right)$$

Value function: $V(k,x) = \inf_u \left\{ \sum_{n=k}^{N-2} u_n^2 + X_{N-1}^2 \right\} : \quad u_n \in [0, X_n]$

DP: $V(k,x) = \inf_{0 \le u \le x} \left(u_k^2 + V(k+1, x-u) \right).$

Backward iteration:
- $V(N-1, x) = x^2.$
- $V(N-2, x) = \inf_{0 \le u \le x} \left(u^2 + (x-u)^2 \right) = \dfrac{x^2}{2}.$ $\quad u = \dfrac{x}{2}.$
- $V(N-3, x) = \inf_{0 \le u \le x} \left(u^2 + \dfrac{(x-u)^2}{2} \right) = \dfrac{x^2}{3}$

$V(\ldots) = \ldots x^2$

$$- \; V(N-k, x) = \frac{x^2}{k} \Rightarrow V(0, x) = \frac{x^2}{N}. \quad \#.$$

Remark: The static approach is better.

Exercise 2.

$$V(k_0) := \sup_{(k_t)_{t=0\cdots\infty}} \sum_{t=0}^{\infty} \beta^t \ln(\underbrace{k_t^{\alpha} - k_{t+1}}_{c_t}).$$

$\alpha \in (0,1)$

$\beta \in (0,1)$

- k_t capital at time t.
- k_t^{α} is the production at time t.
- c_t is the consumption. so that $\boxed{k_{t+1} = k_t^{\alpha} - c_t.}$

$\Longleftrightarrow c_t = k_t^{\alpha} - k_{t+1}.$

$\boxed{k_{t+1} \in [0, k_t^{\alpha})}$

$\Updownarrow$

$\boxed{c_t \in [0, k_t^{\alpha}].}$

- $\ln(c)$ utility function

1. Define. $W(k) := \dfrac{\alpha \ln(k)}{1-\alpha\beta}.$ and prove that $V(k) \leq W(k), \; \forall k > 0.$

Proof: Let $(c_t)_{t\geq 0}$ be a sequence of consumption, then $k_{t+1} = k_t^{\alpha} - c_t \leq k_t^{\alpha}.$

$$\Rightarrow k_1 \leq k_0^{\alpha}, \quad k_2 \leq k_0^{\alpha^2}, \quad \cdots; \quad k_t \leq k_0^{\alpha^t}.$$

$$\Rightarrow c_t \leq k_t^{\alpha} \leq k_0^{\alpha^{t+1}}$$

$$\Rightarrow V(k_0) = \sup_{c} \sum_{t=0}^{+\infty} \beta^t \ln(c_t) \leq \sum_{t=0}^{+\infty} \beta^t \ln\left(k_0^{\alpha^{t+1}}\right).$$

$$= \sum_{t=0}^{+\infty} \alpha (\alpha\beta)^t \ln(k_0).$$

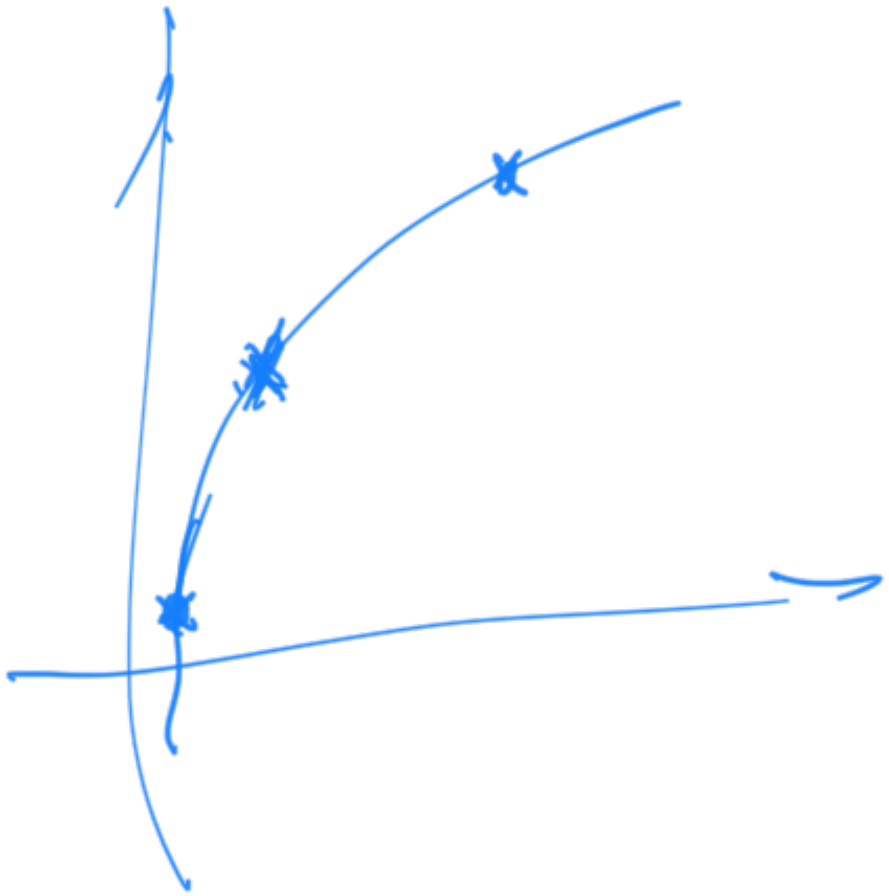
$$= \alpha \ln(k_0) \cdot \sum_{t=0}^{+\infty} (\alpha\beta)^t = \frac{\alpha \ln(k_0)}{1-\alpha\beta} = W(k_0).$$

2. Write down the DP. equation.

$$V(k) = \sup_{c_0 \in [0, k^\alpha]} \left(\ln(c_0) + \beta V(k^\alpha - c_0) \right). =: T(V)(k).$$

3. Compute $T(W)(k)$.

By definition. $T(W)(k) = \sup_{c_0 \in [0, k^\alpha)} \left(\ln(\underbrace{c_0}_{k^\alpha - y}) + \beta \frac{\alpha \ln(\underbrace{k^\alpha - c_0}_{y})}{1-\alpha\beta} \right).$



$$= \sup_{y\in(0,k^\alpha)} \Big(\underbrace{\ln(k^\alpha - y) + \frac{\alpha\beta}{1-\alpha\beta}\ln(y)}_{f(y)} \Big).$$

$$f'(y) = \frac{-1}{k^\alpha - y} + \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{y} = 0.$$

$$\Rightarrow \frac{\alpha\beta}{1-\alpha\beta}\cdot\frac{1}{y} = \frac{-1}{y-k^\alpha} \Rightarrow -\alpha\beta(y-k^\alpha) = y(1-\alpha\beta)$$

$$\Rightarrow y = \alpha\beta k^\alpha.$$

$$\Rightarrow T(W)(k) = \ln\big(k^\alpha(1-\alpha\beta)\big) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta k^\alpha)$$

$$= \ln(k^\alpha) + \ln(1-\alpha\beta) + \frac{\alpha\beta\cdot\ln(\alpha\beta)}{1-\alpha\beta} + \frac{\alpha\beta\cdot\ln(k^\alpha)}{1-\alpha\beta}$$

$$= \underbrace{\frac{1}{1-\alpha\beta}\ln(k^\alpha)} + \underbrace{\ln(1-\alpha\beta) + \frac{\alpha\beta\ln(\alpha\beta)}{1-\alpha\beta}}.$$

$$= W(k) + C, \text{ where } C := \ln(1-\alpha\beta) + \frac{\alpha\beta\ln(\alpha\beta)}{1-\alpha\beta}$$

$1-\alpha\beta$

4: Compute $W^{\infty} \triangleq \lim_{n\to+\infty} T^n(W)$. and show that $W^{\infty} = T(W^{\infty})$.

By 3), $T(W) = W + c$. $\Longrightarrow$ $T^n(W) = W + c + \beta c + \cdots + \beta^n c$.

$\Longrightarrow$ $W^{\infty}(k) = W(k) + c\sum_{n=0}^{+\infty}\beta^n = W(k) + \frac{c}{1-\beta}$.

$\Longrightarrow$ $W^{\infty} = T(W^{\infty})$.

$\Longrightarrow$ Since V is the unique solution of $V = T(V)$

then $V = W^{\infty}$.

5). Show that $C_t^* := (1-\alpha\beta)k_t^{\alpha}$ is the optimal consumption strategy.

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